What's on the mind of air traffic controllers and algebraic geometers?

Abstract

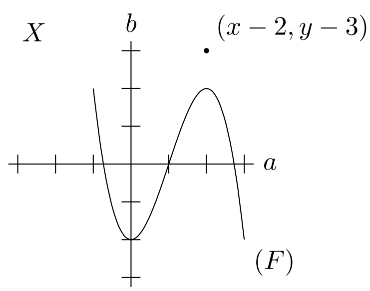
(Subtitled: an introduction to jet spaces.) To any given commutative ring one can associate a topological object known as an affine scheme. The space in question is assigned the Zariski topology, which is in some ways pathological - it is not even Hausdorff. Yet still, jet spaces are a tool which allow us to study analytic behavior, like differential equations, on a scheme. How do we take derivatives and think about ε-neighborhoods outside of the context of metric spaces? In this talk, we find out.

1 Affine Schemes

The objects of consideration are affine schemes. Given a ring R, we can construct a locally ringed space $\operatorname{Spec} R$ which contains two pieces of information:

1. a topological space X. The set is $\{\mathfrak{p} \mid \mathfrak{p} \subseteq R \text{ is a prime ideal}\}$ and we give the set the Zariski topology. We can define the Zariski topology via a basis of open sets, the collection $\mathcal{B} = \{U_f\}$, where we let $U_f = \{\mathfrak{p} \mid f \in R, f \notin \mathfrak{p}\}$ be a distinguished open.

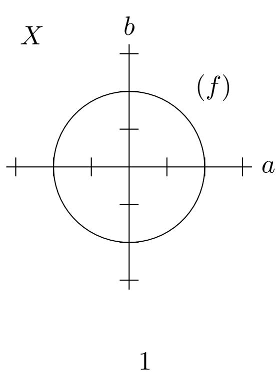
Example 1.1. Let $R = \mathbf{C}[x, y]$. The maximal ideals of R are $(x - a, y - b)$ for $a, b \in \mathbf{C}$. One can show that the prime ideals are those that are maximal, (0), and those of the form (F) for $F \in R$ an irreducible polynomial. As a space, we have



i.e., we should think of both $(x - 2, y - 3)$ and (F) as points in our space. If we let $f = x^2 + y^2 - 4$, then

$$\begin{aligned} U_f &= \{\mathfrak{p} \mid f \notin \mathfrak{p}\} \\ &= X \setminus \{\mathfrak{p} \mid f \in \mathfrak{p}\} \\ &= X \setminus (f), \end{aligned}$$

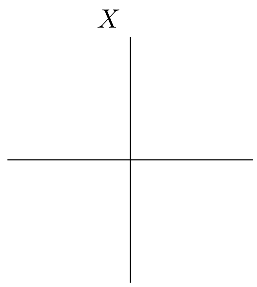
since f is irreducible, hence prime.



Open sets are large; they are essentially complements of zero sets. □

Theorem 1.2. *For $R = \mathbf{C}[x_1, \dots, x_d]/(f_1, \dots, f_s)$, one has $X = \{x \in \mathbf{C}^d \mid f_i(x) = 0$ for all $i\}$.* □

Example 1.3. Let $R = \mathbf{C}[x, y]/(xy)$. The topological space X is

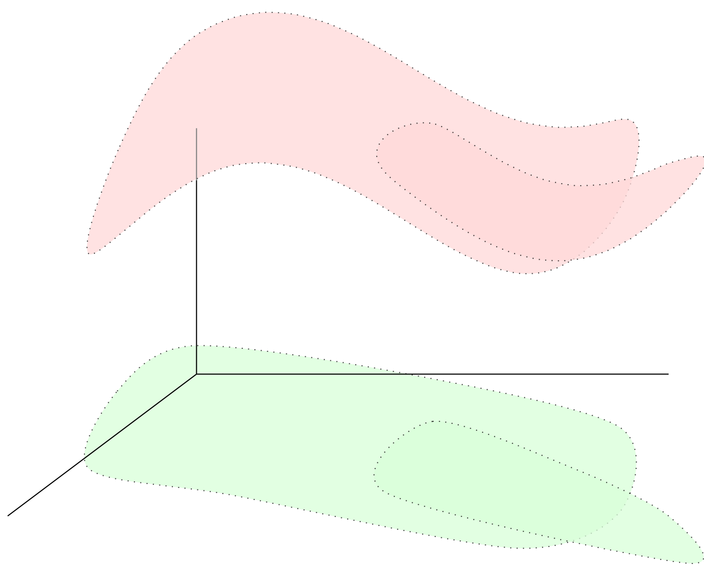


□

2. a sheaf of rings $\mathcal{O}_X$. A presheaf $\mathcal{F}$ is a contravariant functor $\text{Open}(X) \to$ **Rings**; i.e., to each open $U \subseteq X$ we get a ring $\mathcal{F}(U)$, and if $U \subseteq V$, then we get a ring homomorphism $\mathcal{F}(V) \to \mathcal{F}(U)$. We call elements in $\mathcal{F}(U)$ sections.

Example 1.4. The collection of holomorphic functions on a complex manifold X forms a presheaf, since to each open $U \subseteq X$ we get a ring of holomorphic functions $\mathcal{F}(U) = \{f : U \to \mathbf{C} \mid f \text{ is holomorphic}\}$, and if $U \subseteq V$, then we have an assignment $\mathcal{F}(V) \to \mathcal{F}(U)$ given by restriction; a holomorphic function on a large set V is holomorphic on an open subset U of V. □

To go from a presheaf to a sheaf, we impose the gluing condition: given an open cover U_i of X with sections $s_i \in \mathcal{F}(U_i)$, if $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there exists a global section s such that $s|_{U_i} = s_i$ for all i.



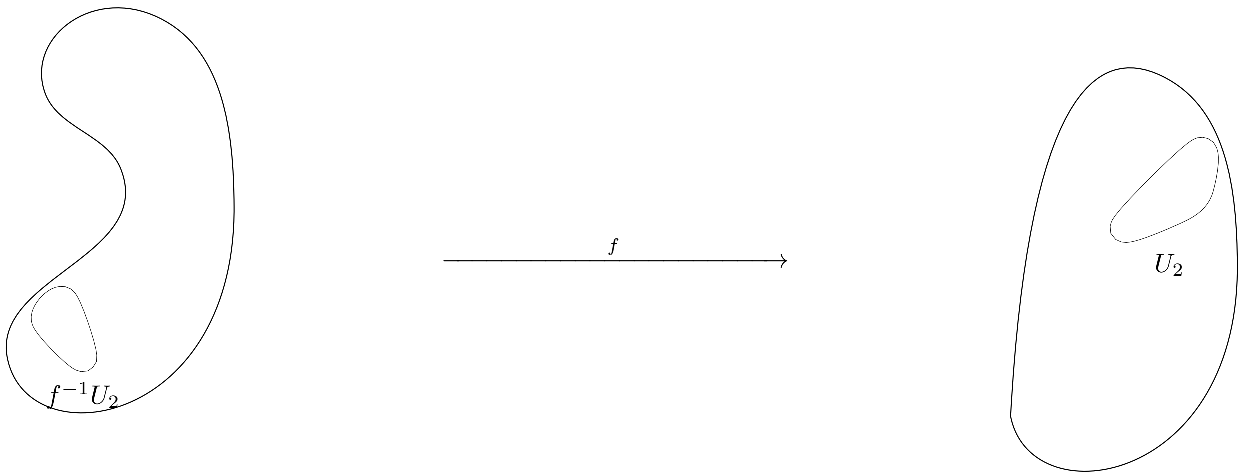
Example 1.5. Holomorphic functions form a sheaf as well, since holomorphic functions glue on open sets, by the identity principle. □

For $\operatorname{Spec} R$, the specific sheaf of rings is called the structure sheaf, and is defined by, for U_f a distinguished open, $\mathcal{O}_X(U_f) = R_f$, the localization of R at f.

Example 1.6. For $R = \mathbf{C}[x, y]/(xy)$, we see that $\operatorname{Spec} R$ is $X = +$ and $\mathcal{O}_X$ is defined via declaring $\mathcal{O}_X(U_f) = \mathbf{C}[x, y]_f/(xy)_f$. □

Notice that for any R, $\mathcal{O}_X(X) = \mathcal{O}_X(U_1) = R_1 = R$. Thus we have a correspondence between rings R and affine schemes $\operatorname{Spec} R$ which has an inverse; given $(X, \mathcal{O}_X)$, $\mathcal{O}_X(X)$ recovers R.

Observe that if we have a map of topological spaces $f : X \to Y$, then we get a pullback map of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$; i.e., for every open subset $U_2 \subseteq Y$, we can produce a map of rings, which is the map $\mathcal{O}_Y(U_2) \to f_*\mathcal{O}_X(U_2) = \mathcal{O}_X(f^{-1}U_2)$.



If we pullback all of Y, we get a map of rings $\mathcal{O}_Y(Y) \to f_*\mathcal{O}_X(Y) = \mathcal{O}_X(f^{-1}Y) = \mathcal{O}_X(X)$. Thus if $\operatorname{Spec} R = (X, \mathcal{O}_X)$ and $\operatorname{Spec} S = (Y, \mathcal{O}_Y)$, then a map of sheaves $\operatorname{Spec} R \to \operatorname{Spec} S$ gives us a map of rings $\mathcal{O}_Y(Y) = S \to \mathcal{O}_X(X) = R$.

Theorem 1.7. *There is a contravariant equivalence of categories between* **AffSch** *and* **Rings**. □

All schemes we will consider will be over $\mathbf{C}$, in the sense that there exists a map $X \to \operatorname{Spec} \mathbf{C}$. By above, that gives a map $\mathbf{C} \to \mathcal{O}_X(X)$; i.e., $\mathcal{O}_X(X)$ is a $\mathbf{C}$-algebra.

Corollary 1.8. *There is a contravariant equivalence of categories between* $\mathbf{AffSch_C}$ *and* $\mathbf{Alg_C}$. □

2 Functors of Points

Goal: we want to see why a scheme X is completely determined (up to isomorphism) by its functor of points, $\operatorname{Hom}_{\mathbf{Sch}}(-, X) : \mathbf{Sch} \to \mathbf{Set}$.

- Why is this called a functor of points? Let's see a simplifying example in **Top**. Let X be a topological space. To see its honest-to-god points, observe that that is equivalent to determining all maps $\{*\} \to X$; i.e., $\operatorname{Hom}_{\mathbf{Top}}(\{*\}, X)$. We can take points to mean something more general, though. We could ask for the "interval-points" of X, which would be $\operatorname{Hom}_{\mathbf{Top}}(I, X)$. We can and do ask for the loops in X, $\operatorname{Hom}_{\mathbf{Top}}(S^1, X)$, i.e., its S^1-valued points, etc. Hence the name functor of points.
- Why should this determine X? Again, think about **Top**. We use such functors all the time already as homeomorphism invariants; i.e., we can tell when two spaces X and Y are different by seeing that they have different fundamental groups $\pi_1(X) = \operatorname{Hom}_{\mathbf{Top}}(S^1, X)/\sim$. If X and Y have the same fundamental groups, we can move to higher homotopy groups $\pi_i(X) = \operatorname{Hom}_{\mathbf{Top}}(S^i, X)/\sim$. If we range over all possible topological-space-valued points of X and Y, either we will find some homeomorphism invariant, or, if we range over *everything* and find no differences, then $X \cong Y$.

Theorem 2.1 (Yoneda Lemma/Corollary)**.** *In a category* $\mathcal{C}$*,* $A \cong B$ *if and only if* $\operatorname{Hom}_{\mathcal{C}}(-, A) \cong \operatorname{Hom}_{\mathcal{C}}(-, B)$. □

Therefore, the functor $\operatorname{Hom}_{\mathbf{Sch}}(-, X)$ determines X. If we plug a scheme Y into $\operatorname{Hom}_{\mathbf{Sch}}(-, X)$, we say it is a Y-valued point of X. If $Y = \operatorname{Spec} S$, then we also say it is an S-valued point of X. Points are also called test objects.

One may ask for the opposite consideration; given a functor, is it the functor of points for some object X? In other words, if F is any functor, does there possibly exist an object X in some category $\mathcal{C}$ such that $F(-) \cong \mathrm{Hom}_{\mathcal{C}}(-, X)$? This is asking for the representability of the functor F. It is an interesting question in general. For us, functors will be representable.

3 Jet Spaces/Arc Spaces

$\operatorname{Spec} R$ is a very algebraic construction. A jet space gives us a way to compute analytic information on $\operatorname{Spec} R$! How?

Definition 3.1. Let X be a scheme. Define the ***n*th jet space of** X, written $J^n X$, to be the scheme whose S-valued points are precisely the $S[t]/t^{n+1}$-valued points of X. That is, $J^n X$ is the representing object of the functor $S \mapsto \mathrm{Hom}_{\mathbf{Sch}}(\operatorname{Spec} S[t]/t^{n+1}, X)$. In other words,

$$\mathrm{Hom}_{\mathbf{Sch}}(\operatorname{Spec} S, J^n X) \cong \mathrm{Hom}_{\mathbf{Sch}}\left(\operatorname{Spec} {}^{S[t]}\!\big/_{t^{n+1}}, X\right).$$

We also define the **arc space of** X, $J^\infty X$, to be the scheme whose S-valued points are the $S[\![t]\!]$-valued points of X; i.e.,

$$\mathrm{Hom}_{\mathbf{Sch}}(\operatorname{Spec} S, J^\infty X) \cong \mathrm{Hom}_{\mathbf{Sch}}(\operatorname{Spec} S[\![t]\!], X).$$

□

Example 3.2 (The dumbest example)**.** Given any X, $J^0 X \cong X$, since $S[t]/t^1 \cong S$. Then apply Yoneda. □

Example 3.3 (A less illuminating example compared to what follows, just to see computation)**.** Consider the affine scheme $X = \operatorname{Spec} \mathbf{C}[x, y]$. What is $J^2 X$? Let S be a test $\mathbf{C}$-algebra. By definition, $J^2 X$ is defined by being the scheme such that

$$\mathrm{Hom}_{\mathbf{Sch}}(\operatorname{Spec} S, J^2 X) \cong \mathrm{Hom}_{\mathbf{Sch}}\left(\operatorname{Spec} {}^{S[t]}\!\big/_{t^3}, X\right).$$

We compute:

$$\mathrm{Hom}_{\mathbf{Sch}_{\mathbf{C}}}\left(\operatorname{Spec} {}^{S[t]}\!\big/_{t^3}, \operatorname{Spec} \mathbf{C}[x, y]\right) \cong \mathrm{Hom}_{\mathbf{Alg}_{\mathbf{C}}}\left(\mathbf{C}[x, y], {}^{S[t]}\!\big/_{t^3}\right)$$

What does an algebra map $\mathbf{C}[x, y] \to S[t]/t^3$ look like? We must determine the image of x and y; they will be

$$\begin{aligned} x &\mapsto a_0 + a_1 t + a_2 t^2 \\ y &\mapsto b_0 + b_1 t + b_2 t^2 \end{aligned}$$

for $a_0, a_1, a_2, b_0, b_1, b_2 \in S$. Thus, to give a map $\mathbf{C}[x, y] \to S[t]/t^3$ is the same as giving a map of $\mathbf{C}$-algebras $\mathbf{C}[a_0, a_1, a_2, b_0, b_1, b_2] \to S$. Therefore

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Alg}_{\mathbf{C}}}\left(\mathbf{C}[x, y], {}^{S[t]}\!\big/_{t^3}\right) &\cong \mathrm{Hom}_{\mathbf{Alg}_{\mathbf{C}}}\left(\mathbf{C}[a_0, a_1, a_2, b_0, b_1, b_2], S\right) \\ &\cong \mathrm{Hom}_{\mathbf{Sch}_{\mathbf{C}}}\left(\operatorname{Spec} S, \operatorname{Spec} \mathbf{C}[a_0, a_1, a_2, b_0, b_1, b_2]\right). \end{aligned}$$

Therefore, $J^2 X \cong \operatorname{Spec} \mathbf{C}[a_0, a_1, a_2, b_0, b_1, b_2]$. □

Example 3.4 (A more illuminating example, to see analysis)**.** Let $Y = \operatorname{Spec} \mathbf{C}[x, y]/(xy)$. What is $J^2 Y$? Let S be a test $\mathbf{C}$-algebra, and we know

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Sch}_{\mathbf{C}}}\left(\operatorname{Spec} S, J^2 Y\right) &\cong \mathrm{Hom}_{\mathbf{Sch}_{\mathbf{C}}}\left(\operatorname{Spec} {}^{S[t]}\!\big/_{t^3}, Y\right) \\ &\cong \mathrm{Hom}_{\mathbf{Alg}_{\mathbf{C}}}\left({}^{\mathbf{C}[x, y]}\!\big/_{(xy)}, {}^{S[t]}\!\big/_{t^3}\right). \end{aligned}$$

Once again, an algebra map must send

$$x \mapsto a_0 + a_1 t + a_2 t^2$$
$$y \mapsto b_0 + b_1 t + b_2 t^2.$$

This time, we also require that the image of the product xy be zero in $S[t]/t^3$. Therefore we have

$$0 = (a_0 + a_1 t + a_2 t^2)(b_0 + b_1 t + b_2 t^2) = a_0 b_0 + (a_0 b_1 + a_1 b_0)t + (a_0 b_2 + a_1 b_1 + a_2 b_0)t^2 + \left(t^3\right);$$

i.e.,

$$\begin{aligned} a_0 b_0 &= 0, \\ a_0 b_1 + a_1 b_0 &= 0, \text{ and} \\ a_0 b_2 + a_1 b_1 + a_2 b_0 &= 0. \end{aligned}$$

Therefore, an algebra map $\mathbf{C}[x, y]/(xy) \to S[t]/t^3$ is equivalent to a map

$$\mathbf{C}[a_0, a_1, a_2, b_0, b_1, b_2] \big/ (a_0 b_0, a_0 b_1 + a_1 b_0, a_0 b_2 + a_1 b_1 + a_2 b_0) \to S,$$

and therefore $J^2 Y \cong \operatorname{Spec} \mathbf{C}[a_0, a_1, a_2, b_0, b_1, b_2]/(a_0 b_0, a_0 b_1 + a_1 b_0, a_0 b_2 + a_1 b_1 + a_2 b_0)$.

Why is this more enlightening? Because if we do a simple change of notation, we have

$$J^2 Y \cong \operatorname{Spec} \mathbf{C}[x, x', x'', y, y', y''] \big/ (xy, xy' + x'y, xy'' + x'y' + x''y).$$

One final change of variables allows us to write

$$J^2 Y \cong \operatorname{Spec} \mathbf{C}[x, x', x'', y, y', y''] \big/ (xy, (xy)', (xy)'').$$

Thus the jet space is indeed giving us information about derivatives! In fact, in the previous example, $J^2 X \cong \mathbf{C}[x, x', x'', y, y', y'']$. □

Theorem 3.5. *If $R = \mathbf{C}[x_\alpha]/(f_\beta)$, then $J^n \operatorname{Spec} R \cong \mathbf{C}[x_\alpha, {x_\alpha}', {x_\alpha}'', \ldots, {x_\alpha}^{(n)}]/(f_\beta, {f_\beta}', {f_\beta}'', \ldots, {f_\beta}^{(n)})$ and $J^\infty \operatorname{Spec} R \cong \mathbf{C}[x_\alpha, {x_\alpha}', {x_\alpha}'', \ldots]/(f_\beta, {f_\beta}', {f_\beta}'', \ldots)$.* □

The main idea is that if X has an S-point, then an $S[t]/t^{n+1}$-point can be thought of as an infinitesimal neighborhood of the point that captures information about tangent spaces (e.g., $S[t]/t^2$, since we can think of infinitesimals in calculus as a small h or ε such that $\varepsilon^2 = 0$). One can run this exact same machinery on any representable functor F to get the **tangent space to a functor**.